

Quasifinite representations of a class of Block type Lie algebras $\mathcal{B}(q)$ ¹

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Abstract. Intrigued by a well-known theorem of Mathieu's on Harish-Chandra modules over the Virasoro algebra, we give an analogous result for a class of Block type Lie algebras $\mathcal{B}(q)$, where the parameter q is a nonzero complex number. We also classify quasifinite irreducible highest weight $\mathcal{B}(q)$ -modules and irreducible $\mathcal{B}(q)$ -modules of the intermediate series. In particular, we obtain that an irreducible $\mathcal{B}(q)$ -module of the intermediate series may be a nontrivial extension of a Vir-module of the intermediate series if q is half of a negative integer, where Vir is a subalgebra of $\mathcal{B}(q)$ isomorphic to the Virasoro algebra.

Key words: Block type Lie algebras; quasifinite modules; highest weight modules; uniform bounded modules; modules of the intermediate series.

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1. Introduction

Since a class of infinite dimensional simple Lie algebras were introduced by Block [1], generalizations of Lie algebras of this type (usually referred to as *Block type Lie algebras*) have been studied by many authors (see, e.g., [2, 8, 12–14, 16–18, 20–22]). Even so, the representation theory for Block type Lie algebras is far from being well developed, except for quasifinite representations of some particular Block type Lie algebras (see, e.g., [12–14, 16]). For example, the author of [12, 13] studied the representations of the Block type Lie algebra \mathcal{B} with basis $\{L_{\alpha,i}, c \mid \alpha, i \in \mathbb{Z}, i \geq -1\}$ over \mathbb{C} and relations

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+1) - \alpha(j+1))L_{\alpha+\beta,i+j} + \alpha\delta_{\alpha+\beta,0}\delta_{i+j,-2}c, \quad [c, L_{\alpha,i}] = 0, \quad (1.1)$$

for $\alpha, \beta \in \mathbb{Z}$, $i, j \geq -1$. The author of [16] studied representations of the Block type Lie algebra $\mathcal{B}(1)$, which can realized as a special case of Block type Lie algebras considered in this paper. The author of [14] presented some results on the classification of quasifinite representations of Lie algebras related to the Virasoro algebra, including some Block type Lie algebras.

In this paper, we study systematically representations of Block type Lie algebras for a class $\mathcal{B}(q)$ (not only for a single algebra) with parameter q being a nonzero complex

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number, where $\mathcal{B}(q)$ has basis $\{L_{\alpha,i}, c \mid \alpha, i \in \mathbb{Z}, i \geq 0\}$ over \mathbb{C} and relations

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q))L_{\alpha+\beta,i+j} + \delta_{\alpha+\beta,0}\delta_{i+j,0}\frac{\alpha^3 - \alpha}{12}c, \quad [c, L_{\alpha,i}] = 0. \quad (1.2)$$

Note that the Lie algebra $\mathcal{B}(0)$ is in fact a half part of the well-known Virasoro-like algebra, and $\mathcal{B}(1)$ is the Block type Lie algebra studied in [16].

The Lie algebra $\mathcal{B}(q)$ has a natural \mathbb{Z} -gradation $\mathcal{B}(q) = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{B}(q)_\alpha$ with

$$\mathcal{B}(q)_\alpha = \text{span}\{L_{\alpha,i} \mid \alpha, i \in \mathbb{Z}, i \geq 0\} \oplus \delta_{\alpha,0}\mathbb{C}c. \quad (1.3)$$

Definition 1.1 (1) A module V over $\mathcal{B}(q)$ is called

- *\mathbb{Z} -graded* if $V = \bigoplus_{\alpha \in \mathbb{Z}} V_\alpha$ and $\mathcal{B}(q)_\alpha V_\beta \subset V_{\alpha+\beta}$ for all α, β ;
- *quasifinite* if it is \mathbb{Z} -graded and $\dim V_\beta < \infty$ for all β ;
- *uniformly bounded* if it is \mathbb{Z} -graded and there is $N \geq 0$ with $\dim V_\beta \leq N$ for all β ;
- *a module of the intermediate series* if it is \mathbb{Z} -graded and $\dim V_\beta \leq 1$ for all β ;
- a *highest* (resp., *lowest*) *weight module* if there exists some $\Lambda \in \mathcal{B}(q)_0^*$ (the dual space of $\mathcal{B}(q)_0$) such that $V = V(\Lambda)$, where $V(\Lambda)$ is a module generated by a *highest* (resp., *lowest*) *weight vector* $v_\Lambda \in V(\Lambda)_0$, i.e., v_Λ satisfies

$$hv_\Lambda = \Lambda(h)v_\Lambda \quad \text{for } h \in \mathcal{B}(q)_0, \quad \text{and } \mathcal{B}(q)_+v_\Lambda = 0 \quad (\text{resp., } \mathcal{B}(q)_-v_\Lambda = 0),$$

where $\mathcal{B}(q)_\pm = \bigoplus_{\pm\alpha > 0} \mathcal{B}(q)_\alpha$ (cf. (2.3)).

(2) A nonzero vector v in a \mathbb{Z} -graded module V is called *singular* or *primitive* if $\mathcal{B}(q)_+v = 0$.

When we study representations of a Lie algebra of this kind, as pointed in [3, 10, 15], we encounter the difficulty that though it is \mathbb{Z} -graded, the graded subspaces are still infinite dimensional, thus the study of quasifinite modules is a nontrivial problem. As stated in [16], an important feature that $\mathcal{B}(q)$ defined in (1.2) is different from \mathcal{B} defined in (1.1) is that $\mathcal{B}(q)$ contains the subalgebra Vir isomorphic to the well-known Virasoro algebra, where

$$\text{Vir} = \text{span}\{L_\alpha, \kappa \mid \alpha \in \mathbb{Z}\}, \quad L_\alpha := q^{-1}L_{\alpha,0}, \quad \kappa := q^{-2}c, \quad (1.4)$$

$$[L_\alpha, L_\beta] = (\beta - \alpha)L_{\alpha+\beta} + \frac{\alpha^3 - \alpha}{12}\delta_{\alpha+\beta,0}\kappa, \quad [\kappa, L_\alpha] = 0. \quad (1.5)$$

The authors of [19] studied structures of $\overline{\mathcal{B}(q)} = \mathcal{B}(q)/\mathbb{C}c$ with q being a positive integer. We point out here that the results [19, Theorems 2.7, 3.1 and 4.1], including automorphism groups, derivation algebras and central extensions, of $\overline{\mathcal{B}(q)}$ for $0 < q \in \mathbb{Z}$ still hold for

$q \in \mathbb{C} \setminus (\frac{1}{2}\mathbb{Z}_- \cup \frac{1}{3}\mathbb{Z}_-)$, since in case $q \in \mathbb{C} \setminus (\frac{1}{2}\mathbb{Z}_- \cup \frac{1}{3}\mathbb{Z}_- \cup \mathbb{Z}_+)$, many coefficients containing q will become invertible. In addition, $\mathcal{B}(q)$'s are distinct from each other for different positive rational number q 's, namely,

$$\mathcal{B}(q_1) \cong \mathcal{B}(q_2) \iff q_1 = q_2 \quad \text{for } q_1, q_2 \in \mathbb{Q}_+^*.$$

Furthermore, for any $1 < q_1 < q_2 \in \mathbb{Z}$ with $q_1 | q_2$, we find the following interesting relations:

$$\mathcal{B}(\varepsilon q_2^{-1}) \subsetneq \mathcal{B}(\varepsilon q_1^{-1}) \subsetneq \mathcal{B}(\varepsilon) \subsetneq \mathcal{B}(\varepsilon q_1) \subsetneq \mathcal{B}(\varepsilon q_2), \quad \text{where } \varepsilon = \pm 1. \quad (1.6)$$

More precisely, $\mathcal{B}(\varepsilon)$ contains the subalgebra with basis $\{q_1^{-1}L_{\alpha, q_1 i} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$ isomorphic to $\mathcal{B}(\varepsilon q_1^{-1})$, and $\mathcal{B}(\varepsilon q_2)$ contains the subalgebra with basis $\{q_2^{-1}L_{\alpha, q_2 i} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$ isomorphic to $\mathcal{B}(\varepsilon)$.

Moreover, $\mathcal{B}(q)$, $\mathcal{B}(q^{-1})$ with $0 < q \in \mathbb{Z}$ are related to the well-known W -infinity Lie algebra \mathcal{W}_∞ in the following way: Recall that the W -infinity Lie algebra $\mathcal{W}_{1+\infty}$ is defined to be the universal central extension of infinite dimensional Lie algebra of differential operators on the circle, which has basis $\{x^\alpha D^i, c \mid \alpha \in \mathbb{Z}, i \geq 0\}$ with $D = \frac{d}{dx}$, and relations

$$[x^\alpha D^i, x^\beta D^j] = x^{\alpha+\beta}((D+\beta)^i D^j - D^i (D+\alpha)^j) + \delta_{\alpha+\beta, 0}(-1)^i i! j! \binom{\alpha+i}{i+j+1} c.$$

Then the W -infinity algebra \mathcal{W}_∞ , the universal central extension of infinite dimensional Lie algebra of differential operators on the circle of degree at least one, is simply the subalgebra of $\mathcal{W}_{1+\infty}$ spanned by $\{x^\alpha D^i, c \mid \alpha \in \mathbb{Z}, i \geq 1\}$. If we define a natural filtration of \mathcal{W}_∞ by

$$\begin{aligned} \{0\} &= (\mathcal{W}_\infty)_{[-2]} \subset (\mathcal{W}_\infty)_{[-1]} \subset \cdots \subset \mathcal{W}_\infty, \quad \text{where} \\ (\mathcal{W}_\infty)_{[-1]} &= \mathbb{C}c, \quad (\mathcal{W}_\infty)_{[n]} = \text{span}\{x^\alpha D^i, c \mid \alpha \in \mathbb{Z}, 1 \leq i \leq n+1\} \text{ for } n \geq 0, \end{aligned}$$

then $\mathcal{B}(1)$ is simply the associated graded Lie algebra of the filtered Lie algebra \mathcal{W}_∞ . So roughly speaking, $\mathcal{B}(q)$ contains (reps., $\mathcal{B}(q^{-1})$ lies in) the associated graded Lie algebra of \mathcal{W}_∞ by the fact (1.6). As stated in [10, 12, 15], the W -infinity algebras arise naturally in various physical theories, such as conformal field theory, the theory of the quantum Hall effect, etc.; among them the \mathcal{W}_∞ algebra and $\mathcal{W}_{1+\infty}$ algebra, of interest to both mathematicians and physicists, have received intensive studies in the literature. Due to the importance of the W -infinity algebra \mathcal{W}_∞ , motivated by (1.6), it is very natural to post the following question, which seems to be interesting to us.

Question 1.2 *Do there exist Lie algebras $\mathcal{W}(q)$ for $q \in \mathbb{Z}_+$ or $q^{-1} \in \mathbb{Z}_+$ such that $\mathcal{W}(1) = \mathcal{W}_\infty$ and $\mathcal{W}(q_2^{-1}) \subsetneq \mathcal{W}(q_1^{-1}) \subsetneq \mathcal{W}(1) \subsetneq \mathcal{W}(q_1) \subsetneq \mathcal{W}(q_2)$ for $1 < q_1 < q_2 \in \mathbb{Z}$ and $q_1 | q_2$.*

Because of the facts stated in the statements before (1.4), one may expect richer results in the representation theory of $\mathcal{B}(q)$. Motivated by a well-known result of Mathieu's in [6] (see also [7, 9]), it is very natural to consider the classification of quasifinite irreducible $\mathcal{B}(q)$ -modules. Our first main result is the following theorem (the analogous results to this theorem for the Virasoro algebra, higher rank Virasoro algebras, W -infinity algebras, and some Block type Lie algebras were obtained in [5, 6, 10–13, 15, 16]).

Theorem 1.3 *A quasifinite irreducible $\mathcal{B}(q)$ -module is either a highest/lowest weight module, or a uniformly bounded module.*

Our second main result is to give a classification of quasifinite irreducible highest weight modules. To state the result, we need to introduce the generating series: For any function $\Lambda \in \mathcal{B}(q)_0^*$ (the dual of $\mathcal{B}(q)_0$), we set *labels* $\Lambda_i = \Lambda(L_{0,i})$ for $i \geq 0$, and define the following *generating series* with variable z ,

$$\Delta_\Lambda(z, q) = 2q \sum_{i=0}^{\infty} \frac{z^i}{i!} \Lambda_i + \sum_{i=0}^{\infty} \frac{z^{i+1}}{i!} \Lambda_{i+1} = \Lambda((2q + zt)t^q e^{zt}). \quad (1.7)$$

We would like to mention that our generating series $\Delta_\Lambda(z, 1)$ corresponding to the Lie algebra $\mathcal{B}(1)$ is different from that given in [16]. Then our second main result below also recovers those stated in [16] for quasifinite irreducible $\mathcal{B}(1)$ -modules.

Theorem 1.4 *Let $L(\Lambda)$ be an irreducible highest weight module over $\mathcal{B}(q)$ with highest weight $\Lambda \in \mathcal{B}(q)_0^*$. Then $L(\Lambda)$ is quasifinite if and only if $\Delta_\Lambda(z, q)$ is a quasipolynomial.*

Our final main result is to give a classification of irreducible modules of the intermediate series. To state the result, let us recall (e.g., [6, 7, 9, 11]) that an indecomposable module of the intermediate series over Vir is one of $A_{a,b}$, A_a , B_a , $a, b \in \mathbb{C}$, or their subquotients, where $A_{a,b}$, A_a , B_a all have a basis $\{v_\mu \mid \mu \in \mathbb{Z}\}$ with the trivial action of c and

$$A_{a,b} : L_{\alpha,0} v_\mu = q(a + \mu + b\alpha) v_{\alpha+\mu}, \quad (1.8)$$

$$A_a : L_{\alpha,0} v_\mu = q(\mu + \alpha) v_{\alpha+\mu} \ (\mu \neq 0), \quad L_{\alpha,0} v_0 = q\alpha(a + \alpha) v_\alpha, \quad (1.9)$$

$$B_a : L_{\alpha,0} v_\mu = q\mu v_{\alpha+\mu} \ (\mu \neq -\alpha), \quad L_{\alpha,0} v_{-\alpha} = -q\alpha(a + \alpha) v_0, \quad (1.10)$$

for $\alpha, \mu \in \mathbb{Z}$ (note that we have the factor “ q ” on the right-hand sides because of notation (1.4)). We use $A'_{0,1}$ to denote the nontrivial subquotients of $A_{0,1}$. Then a nontrivial irreducible Vir-module of the intermediate series is isomorphic to either $A'_{0,1}$ or $A_{a,b}$ ($a \notin \mathbb{Z}$ or $b \neq 0, 1$) with the trivial action of c and

$$A'_{0,1} = \text{span}\{v_\mu \mid \mu \in \mathbb{Z}^*\} : L_{\alpha,0}v_\mu = q(\mu + \alpha)v_{\alpha+\mu}, \quad (1.11)$$

$$A_{a,b} = \text{span}\{v_\mu \mid \mu \in \mathbb{Z}\} : L_{\alpha,0}v_\mu = q(a + \mu + b\alpha)v_{\alpha+\mu}. \quad (1.12)$$

Obviously $A'_{0,1}$ or $A_{a,b}$ is also an irreducible $\mathcal{B}(q)$ -module of the intermediate series (still denoted by $A'_{0,1}$ or $A_{a,b}$) by extending the actions of $L_{\alpha,i}$ with $i \geq 1$ trivially, namely

$$L_{\alpha,i}v_\mu = 0 \quad \text{for } \alpha, i \in \mathbb{Z}, i \geq 1. \quad (1.13)$$

If $q \in \frac{1}{2}\mathbb{Z}_-^*$, for any $s \in \mathbb{C}$, by replacing the actions (1.13) by

$$L_{\alpha,i}v_\mu = \begin{cases} sv_\mu & \text{if } (\alpha, i) = (0, -2q), \\ 0 & \text{if } i \geq 1 \text{ and } (\alpha, i) \neq (0, -2q), \end{cases} \quad (1.14)$$

we obtain an irreducible $\mathcal{B}(q)$ -module of the intermediate series, denoted by $A'_{0,1}(s)$ or $A_{a,b}(s)$. Furthermore, if $q = -1$, for any $s, t \in \mathbb{C}$, the Vir-module $A_{a,b}$ can be defined as a $\mathcal{B}(-1)$ -module, denoted by $A_{a,b}(s, t)$, by replacing (1.13) by

$$L_{\alpha,i}v_\mu = \begin{cases} sv_\mu & \text{if } (\alpha, i) = (0, 2), \\ tv_{\alpha+\mu} & \text{if } i = 1, \\ 0 & \text{if } i \geq 2 \text{ and } (\alpha, i) \neq (0, 2). \end{cases} \quad (1.15)$$

Now we can state our final main result below.

Theorem 1.5 *Let V be an irreducible $\mathcal{B}(q)$ -module of the intermediate series such that it is nontrivial as a Vir-module.*

- (1) *If $q \notin \frac{1}{2}\mathbb{Z}_-^*$, then $V \cong A'_{0,1}$ or $A_{a,b}$ ($a \notin \mathbb{Z}$ or $b \neq 0, 1$).*
- (2) *If $q \in \frac{1}{2}\mathbb{Z}_-^* \setminus \{-1\}$, then $V \cong A'_{0,1}(s)$ or $A_{a,b}(s)$ ($a \notin \mathbb{Z}$ or $b \neq 0, 1$), $s \in \mathbb{C}$.*
- (3) *If $q = -1$, then $V \cong A'_{0,1}(s)$ or $A_{a,b}(s, t)$ ($a \notin \mathbb{Z}$ or $b \neq 0, 1$), $s, t \in \mathbb{C}$.*

Thus in particular, one sees that an irreducible $\mathcal{B}(q)$ -module of the intermediate series for $q \in \frac{1}{2}\mathbb{Z}_-^*$, different from others, can be a nontrivial extension of a Vir-module of the intermediate series.

Based on Theorem 1.3 and results stated in [7] (see also Proposition 4.1), it is very natural to conjecture that an irreducible uniformly bounded $\mathcal{B}(q)$ -module is a module of the intermediate series. Namely,

Conjecture 1.6 *A quasifinite irreducible $\mathcal{B}(q)$ -module is either a highest/lowest weight module, or a module of the intermediate series.*

Throughout the paper, q is always assumed to be a fixed number in \mathbb{C}^* . We use \mathbb{C} , \mathbb{C}^* , \mathbb{Z} , \mathbb{Z}^* , \mathbb{Z}_+ , \mathbb{Z}_+^* , \mathbb{Z}_- and \mathbb{Z}_-^* to denote respectively the sets of complex, nonzero complex numbers, integers, nonzero, nonnegative, positive, nonpositive and negative integers.

2. Proof of Theorem 1.3

We can realize the Lie algebra $\mathcal{B}(q)$ in the space $\mathbb{C}[x, x^{-1}] \otimes t^q \mathbb{C}[t] \oplus \mathbb{C}c$ with the bracket

$$[x^\alpha f(t), x^\beta g(t)] = x^{\alpha+\beta} t^{1-q} (\beta f'(t)g(t) - \alpha f(t)g'(t)) + \delta_{\alpha+\beta,0} \frac{\alpha^3 - \alpha}{12} \text{Res}_t(t^{-2q-1} f(t)g(t))c, \quad (2.1)$$

for $\alpha, \beta \in \mathbb{Z}$ and $f(t), g(t) \in t^q \mathbb{C}[t]$, where the prime stands for the derivative $\frac{d}{dt}$, and $\text{Res}_t f(t)$ stands for the *residue* of $f(t)$, namely the coefficient of t^{-1} in $f(t)$. We always denote

$$L_{\alpha,i} = x^\alpha t^{q+i}. \quad (2.2)$$

Thus (2.1) is equivalent to (1.2). Using the gradation (1.3), we introduce the following notations for $\beta, \gamma \in \mathbb{Z}$,

$$\mathcal{B}(q)_{[\beta,\gamma]} = \sum_{\beta \leq \alpha \leq \gamma} \mathcal{B}(q)_\alpha,$$

and similarly for $\mathcal{B}(q)_{[\beta,+\infty)}$, $\mathcal{B}(q)_{[\beta,\gamma)}$ and so on. Putting $\mathcal{B}(q)_\pm = \oplus_{\pm\alpha>0} \mathcal{B}(q)_\alpha$, we have the following triangular decomposition:

$$\mathcal{B}(q) = \mathcal{B}(q)_- \oplus \mathcal{B}(q)_0 \oplus \mathcal{B}(q)_+. \quad (2.3)$$

Note that $\mathcal{B}(q)_0 = t^q \mathbb{C}[t] \oplus \mathbb{C}c$ is an infinite dimensional commutative subalgebra of $\mathcal{B}(q)$ (but not a Cartan subalgebra).

Now suppose $V = \oplus_{\mu \in \mathbb{Z}} V_\mu$ is a quasifinite $\mathcal{B}(q)$ -module. Taking $\mu_0 \in \mathbb{Z}^*$, since $c|_{V_{\mu_0}}$ (the action of c on V_{μ_0}) and $t^{q+i}|_{V_{\mu_0}}$ for $i \in \mathbb{Z}_+$ are linear transformations of the finite dimensional subspace V_{μ_0} , there exists big enough fixed integer p_0 such that the operators $c|_{V_{\mu_0}}, t^q|_{V_{\mu_0}}, \dots, t^{q+p-1}|_{V_{\mu_0}}$ are linear dependent for all $p \geq p_0$. Therefore, for any $p \geq p_0$, there exists $f_p(t) \in \mathcal{B}(q)_0$ of degree $q+p-1$ and $m_p \in \mathbb{C}$ such that

$$(f_p(t) + m_p c)v = 0 \quad \text{for } v \in V_{\mu_0}. \quad (2.4)$$

Define the Lie subalgebra $\mathcal{L}(q, \mu_0)$ of $\mathcal{B}(q)$ as follows

$$\mathcal{L}(q, \mu_0) = \begin{cases} \langle x^{-\mu_0} t^q, x^{-\mu_0} t^{q+1}, x^{-\mu_0} t^{q+2}, x^{-\mu_0+1} t^q, f_p(t) + m_p c \mid p \geq p_0 \rangle & \text{if } \mu_0 \leq -1, \\ \langle x^{-\mu_0} t^q, x^{-\mu_0} t^{q+1}, x^{-\mu_0} t^{q+2}, x^{-\mu_0-1} t^q, f_p(t) + m_p c \mid p \geq p_0 \rangle & \text{if } \mu_0 \geq 1, \end{cases}$$

where the angle bracket $\langle \cdot, \cdot \rangle$ stands for “the Lie subalgebra generated by”.

Lemma 2.1 *For any $s \geq 1$, and fixed $\mu_0 \in \mathbb{Z}^*$, we have*

- (1) *if $\mu_0 \leq -1$, then there exists $\alpha_s \in \mathbb{Z}_+^*$ such that $x^\alpha t^{q+s-1} \in \mathcal{L}(q, \mu_0)$ for all $\alpha \geq \alpha_s$;*
- (2) *if $\mu_0 \geq 1$, then there exists $\alpha_s \in \mathbb{Z}_-^*$ such that $x^\alpha t^{q+s-1} \in \mathcal{L}(q, \mu_0)$ for all $\alpha \leq \alpha_s$.*

Proof. We only prove part (1) by induction on s (part (2) can be proved similarly). In case $s = 1$, Remark 2.2 below shows that, for any integer $\alpha \geq (1 - \mu_0)^2$, there exist two positive integers k_1, k_2 such that

$$\alpha = k_1(1 - \mu_0) - k_2\mu_0. \quad (2.5)$$

Letting $z_1 = x^{-\mu_0+1}t^q$, $z_2 = x^{-\mu_0}t^q$, using (2.1) and by induction on k_1, k_2 , we obtain

$$\text{ad}_{z_2}^{k_2-1} \text{ad}_{z_1}^{k_1}(z_2) = q^{k_1+k_2-1} \prod_{i=1}^{k_1} (-(i-1)\mu_0 + i - 2) \prod_{j=1}^{k_2-1} (-(k_1+j-1)\mu_0 + k_1) x^\alpha t^q. \quad (2.6)$$

Note that the coefficient of $x^\alpha t^q$ on the right-hand side of (2.6) is nonzero. Hence, $x^\alpha t^q \in \mathcal{L}(q, \mu_0)$. Now suppose $s > 1$, and inductively assume that there exists an integer α_{s-1} such that $x^\alpha t^{q+s-2} \in \mathcal{L}(q, \mu_0)$ for $\alpha \geq \alpha_{s-1}$. We denote $r_{\alpha,q} = \alpha$ if $s = 3$, $q = -1$, or else $r_{\alpha,q} = \mu_0(2q + s - 1) + \alpha(q + 1)$. We can always choose big enough α'_s such that $r_{\alpha,q} \neq 0$ whenever $\alpha \geq \alpha'_s$. Now take $\alpha_s = \max \{\alpha_{s-1} - \mu_0, \alpha'_s\}$, then for all $\alpha \geq \alpha_s$, we have

$$x^\alpha t^{q+s-1} = \begin{cases} -\frac{1}{r_{\alpha,q}} [x^{\alpha+\mu_0} t^{q+s-3}, x^{-\mu_0} t^{q+2}] & \text{if } s = 3, q = -1, \\ -\frac{1}{r_{\alpha,q}} [x^{\alpha+\mu_0} t^{q+s-2}, x^{-\mu_0} t^{q+1}] & \text{else,} \end{cases}$$

which shows $x^\alpha t^{q+s-1} \in \mathcal{L}(q, \mu_0)$. Part (1) is proved. \square

Remark 2.2 The lower bound $(1 - \mu_0)^2$ of α for case $s = 1$ in the above lemma is more precise than that for a quasifinite $\mathcal{B}(1)$ -module given in [16], which can be deduced as follows: For any $\alpha \geq (1 - \mu_0)^2$, we denote $k_0 = [\frac{\alpha}{1-\mu_0}]$ (the integral part of $\frac{\alpha}{1-\mu_0}$). Then $k_0 \geq 1 - \mu_0$, and two integers

$$\begin{aligned} k_1 &:= \alpha + (k_0 + 1)\mu_0 \geq k_0(1 - \mu_0) + (k_0 + 1)\mu_0 = k_0 + \mu_0 \geq 1, \\ k_2 &:= (k_0 + 1)(1 - \mu_0) - \alpha > 0, \end{aligned}$$

satisfy (2.5).

Lemma 2.3 *Let $V = \oplus_{\mu \in \mathbb{Z}} V_\mu$ be a quasifinite irreducible $\mathcal{B}(q)$ -module.*

- (1) *If $\mu_0 \leq -2$, and there exists $0 \neq v_0 \in V_{\mu_0}$ satisfying $\mathcal{B}(q)_{[\alpha, +\infty)} v_0 = 0$ for some $\alpha > 0$, then V has a highest weight vector.*
- (2) *If $\mu_0 \geq 2$, and there exists $0 \neq v_0 \in V_{\mu_0}$ satisfying $\mathcal{B}(q)_{(-\infty, \alpha]} v_0 = 0$ for some $\alpha < 0$, then V has a lowest weight vector.*

One can prove this lemma in a similar way as in [10, 12, 16], and the details are omitted.

Proof of Theorem 1.3. Assume that $V = \oplus_{\mu \in \mathbb{Z}} V_\mu$ is a quasifinite irreducible $\mathcal{B}(q)$ -module without highest and lowest weight vectors. We should prove that

$$\dim V_\mu \leq \begin{cases} 3\dim V_0 + \dim V_1 & \text{if } \mu \leq -2, \\ 3\dim V_0 + \dim V_{-1} & \text{if } \mu \geq 2. \end{cases} \quad (2.7)$$

For fixed $\mu_0 \leq -2$, we claim that the following linear map is injective:

$$\Theta_{\mu_0}^- = \left(x^{-\mu_0} t^q \oplus x^{-\mu_0} t^{q+1} \oplus x^{-\mu_0} t^{q+2} \oplus x^{-\mu_0+1} t^q \right) \Big|_{V_{\mu_0}} : V_{\mu_0} \rightarrow V_0 \oplus V_0 \oplus V_0 \oplus V_1.$$

Otherwise there exists $0 \neq v_0 \in V_{\mu_0}$ such that $\Theta_{\mu_0}^-(v_0) = 0$, which implies that $x^{-\mu_0} t^q$, $x^{-\mu_0} t^{q+1}$, $x^{-\mu_0} t^{q+2}$ and $x^{-\mu_0+1} t^q$ take v_0 to zero. On the other hand, $(f_p(t) + m_p c)v_0 = 0$ for $p \geq p_0$ by (2.4). Hence, by definition,

$$\mathcal{L}(q, \mu_0)v_0 = 0. \quad (2.8)$$

Applying Lemma 2.1(1), for any $1 \leq p < p_0$, there exists some positive integer α_p such that $x^\alpha t^{q+p-1} \in \mathcal{L}(q, \mu_0)$ for $\alpha \geq \alpha_p$. Denote $\Gamma = \max\{\alpha_1, \alpha_2, \dots, \alpha_{p_0-1}\}$. Then $x^\alpha t^{q+p-1} \in \mathcal{L}(q, \mu_0)$ for $1 \leq p < p_0$, $\alpha \geq \Gamma$. Furthermore, for $p \geq p_0$, $\alpha \geq \Gamma$, we have

$$x^\alpha t f'_p(t) = -\frac{1}{\alpha} [x^\alpha t^q, f_p(t) + m_p c] \in \mathcal{L}(q, \mu_0). \quad (2.9)$$

Taking $p = p_0 + i$ in (2.9), noting that $\deg f_p(t) = q + p - 1$, we have $x^\alpha t^{q+p_0+i-1} \in \mathcal{L}(q, \mu_0)$ for $\alpha \geq \Gamma$, $i \in \mathbb{Z}_+$. Therefore $x^\alpha t^{q+p-1} \in \mathcal{L}(q, \mu_0)$ for $p \geq 1$, $\alpha \geq \Gamma$, namely,

$$\mathcal{B}(q)_{[\Gamma, +\infty)} \subseteq \mathcal{L}(q, \mu_0). \quad (2.10)$$

By (2.8) and (2.10), $\mathcal{B}(q)_{[\Gamma, +\infty)} v_0 = 0$. Then Lemma 2.3(1) shows V has a highest weight vector, which contradicts our assumption. Thus the map $\Theta_{\mu_0}^-$ is injective, which implies $\dim V_\mu \leq 3\dim V_0 + \dim V_1$ if $\mu \leq -2$.

Similarly, one can derive $\dim V_\mu \leq 3\dim V_0 + \dim V_{-1}$ if $\mu \geq 2$ by Lemma 2.1(2) and Lemma 2.3(2). Denote $N = \max\{3\dim V_0 + \dim V_1, 3\dim V_0 + \dim V_{-1}\}$. Then $\dim V_\mu \leq N$ for $\mu \in \mathbb{Z}$, namely V is a uniformly bounded $\mathcal{B}(q)$ -module. This completes the proof. \square

3. Quasifinite highest weight modules

In this section, we start with general settings on parabolic subalgebras of \mathbb{Z} -graded Lie algebra. Following [3, 4, 12, 13], after giving some descriptions of parabolic subalgebras of $\mathcal{B}(q)$, we use the results to characterize the irreducible quasifinite highest weight $\mathcal{B}(q)$ -modules by generating series.

Definition 3.1 Let $\mathcal{L} = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{L}_\alpha$ be a \mathbb{Z} -graded Lie algebra.

- (1) A subalgebra \mathcal{P} of \mathcal{L} is called *parabolic* if it contains $\mathcal{L}_0 \oplus \mathcal{L}_+$ as a proper subalgebras, namely, $\mathcal{P} = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{P}_\alpha$ with $\mathcal{P}_\alpha = \mathcal{L}_\alpha$ for $\alpha \geq 0$, and $\mathcal{P}_\alpha \neq \{0\}$ for some $\alpha < 0$.
- (2) Given $0 \neq a \in \mathcal{L}_{-1}$, we define a parabolic subalgebra $\mathcal{P}(a) = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{P}(a)_\alpha$ of \mathcal{L} as follows:

$$\mathcal{P}(a)_\alpha = \begin{cases} \mathcal{L}_\alpha & \text{if } \alpha \geq 0, \\ \text{span}\{[\dots, [\mathcal{L}_0, [\mathcal{L}_0, a]] \dots]\} & \text{if } \alpha = -1, \\ [\mathcal{P}(a)_{-1}, \mathcal{P}(a)_{\alpha+1}] & \text{if } \alpha \leq -2. \end{cases} \quad (3.1)$$

- (3) A parabolic subalgebra \mathcal{P} is called *nondegenerate* if \mathcal{P}_α has finite codimension in \mathcal{L}_α for all $\alpha < 0$.
- (4) A nonzero element $a \in \mathcal{L}_{-1}$ is called *nondegenerate* if $\mathcal{P}(a)$ is nondegenerate.

A *Verma module* over $\mathcal{B}(q)$ is defined as the induced module

$$M(\Lambda) = U(\mathcal{B}(q)) \otimes_{U(\mathcal{B}(q)_0 \oplus \mathcal{B}(q)_+)} \mathbb{C}v_\Lambda \quad \text{for } \Lambda \in \mathcal{B}(q)_0^*,$$

where $\mathbb{C}v_\Lambda$ is the one-dimensional $\mathcal{B}(q)_0 \oplus \mathcal{B}(q)_+$ -module given by $(h+n)(v_\Lambda) = \Lambda(h)v_\Lambda$ for $h \in \mathcal{B}(q)_0, n \in \mathcal{B}(q)_+$ (cf. (2.3)). Here and further $U(\mathcal{L})$ stands for the universal enveloping algebra of a Lie algebra \mathcal{L} . Then any highest weight module $V(\Lambda)$ is a quotient module of $M(\Lambda)$ and the irreducible highest weight module $L(\Lambda)$ is the quotient of $M(\Lambda)$ by the maximal proper \mathbb{Z} -graded submodule.

Define a parabolic subalgebra $\mathcal{P}(q, a) = \bigoplus_{\alpha \in \mathbb{Z}} \mathcal{P}(q, a)_\alpha$ of $\mathcal{B}(q)$ as in (3.1), where $0 \neq a \in \mathcal{B}(q)_{-1}$. By [3, Lemma 2.2], $\mathcal{P}(q, a)$ is the minimal parabolic subalgebra containing a and

$$\mathcal{B}(q, a)_0 := \mathcal{B}(q)_0 \cap [\mathcal{P}(q, a), \mathcal{P}(q, a)] = [a, \mathcal{B}(q)_1].$$

Write $a = x^{-1}f(t)$, then we have $[a, xg(t)] = [x^{-1}f(t), xg(t)] = (f'(t)g(t) + f(t)g'(t))t^{1-q}$ for $g(t) \in t^q\mathbb{C}[t]$, which implies

$$\mathcal{B}(q, a)_0 = \text{span}\{(f(t)g(t))'t^{1-q} \mid g(t) \in t^q\mathbb{C}[t]\}. \quad (3.2)$$

Let $\Lambda \in \mathcal{B}(q)_0^*$ be such that $\Lambda|_{\mathcal{B}(q)_0 \cap [\mathcal{P}(q), \mathcal{P}(q)]} = 0$. Then the $\mathcal{B}(q)_0 \oplus \mathcal{B}(q)_+$ -module $\mathbb{C}v_\Lambda$ can be extended to be a $\mathcal{P}(q)$ -module by letting $\mathcal{P}(q)_\alpha$ take v_Λ to zero for $\alpha < 0$. We construct the following highest weight $\mathcal{B}(q)$ -module

$$M(\mathcal{P}(q), \Lambda) = U(\mathcal{B}(q)) \otimes_{U(\mathcal{P}(q))} \mathbb{C}v_\Lambda,$$

which is called the *generalized Verma module*.

Lemma 3.2 *Let $\mathcal{P}(q) = \oplus_{\alpha \in \mathbb{Z}} \mathcal{P}(q)_\alpha$ be a parabolic subalgebra of $\mathcal{B}(q)$.*

- (1) *There exists an nonzero element $0 \neq a \in \mathcal{B}(q)_{-1}$ such that $\mathcal{P}(q, a) \subseteq \mathcal{P}(q)$.*
- (2) *For any $\alpha < 0$, the subspace $\mathcal{P}(q)_\alpha$ is nontrivial, and has finite codimension in $\mathcal{B}(q)_\alpha$.*
- (3) *$\mathcal{P}(q)$ is nondegenerate, and any nonzero element $0 \neq a \in \mathcal{B}(q)_{-1}$ is nondegenerate.*

Proof. (1) By definition, there exists at least one $\alpha < 0$ such that $\mathcal{P}(q)_\alpha \neq \{0\}$. We claim that $\mathcal{P}(q)_{\alpha+1} \neq \{0\}$ if $\alpha \leq -2$. Otherwise $[\mathcal{P}(q)_\alpha, \mathcal{B}(q)_1] = 0$. Since $\alpha < 0$, we can easily choose some positive integer j_0 such that $k_\alpha = (q+i) - \alpha(q+j_0) \neq 0$ for $i \in \mathbb{Z}_+$. Taking any $0 \neq b = \sum_{i \in I} b_i x^\alpha t^{q+i} \in \mathcal{P}(q)_\alpha$, where I is a finite subset of \mathbb{Z}_+ and $b_i \in \mathbb{C}$, we have

$$0 = [b, xt^{q+j_0}] = \left[\sum_{i \in I} b_i x^\alpha t^{q+i}, xt^{q+j_0} \right] = \sum_{i \in I} b_i k_\alpha x^{\alpha+1} t^{q+i+j_0}, \quad (3.3)$$

which implies $b_i = 0$ for $i \in I$, i.e., $b = 0$, a contradiction. This proves the claim. Therefore $\mathcal{P}(q)_{-1} \neq \{0\}$ by induction. Taking any nonzero element $a \in \mathcal{P}(q)_{-1}$, we have $\mathcal{P}(q, a) \subseteq \mathcal{P}(q)$ by the minimality of $\mathcal{P}(q, a)$.

(2) We shall use induction on $\alpha < 0$ to show $\mathcal{P}(q)_\alpha \neq \{0\}$. The case $\alpha = -1$ is proved in (1). Now suppose $\mathcal{P}(q)_\alpha \neq \{0\}$ for some $\alpha \leq -1$. For $0 \neq x^\alpha f(t) \in \mathcal{P}(q)_\alpha$, we have

$$x^\alpha f(t)t^i = \frac{1}{\alpha(q+i)} [t^{q+i}, x^\alpha f(t)] \in \mathcal{P}(q)_\alpha \quad \text{if } i \neq -q. \quad (3.4)$$

Let $z_1 := x^\alpha f_1(t) \in \mathcal{P}(q)_\alpha$ and $z_2 := x^{-1}f_2(t) \in \mathcal{P}(q)_{-1}$ be any nonzero elements. Then $z_3 := [z_1, z_2] \in \mathcal{P}(q)_{\alpha-1}$. If $q \neq -1$, then (3.4) with $i = 1$ implies $z_3 t \in \mathcal{P}(q)_{\alpha-1}$, $z_2 t \in \mathcal{P}(q)_{-1}$, and so

$$\alpha x^{\alpha-1} f_1(t) f_2(t) t^{1-q} = z_3 t - [z_1, z_2 t] \in \mathcal{P}(q)_{\alpha-1},$$

which is clearly a nonzero element. If $q = -1$, then (3.4) with $i = 2$ implies $z_3 t^2 \in \mathcal{P}(q)_{\alpha-1}$, $z_2 t^2 \in \mathcal{P}(q)_{-1}$, and so

$$2\alpha x^{\alpha-1} f_1(t) f_2(t) t^3 = z_3 t^2 - [z_1, z_2 t^2] \in \mathcal{P}(q)_{\alpha-1},$$

which is clearly a nonzero element. Thus by induction, $\mathcal{P}(q)_\alpha \neq \{0\}$ for all $\alpha < 0$. This together with (3.4) immediately implies that $\mathcal{P}(q)_\alpha$ has finite codimension in $\mathcal{B}(q)_\alpha$.

(3) By definition, $\mathcal{P}(q)$ is nondegenerate by (2). In particular, $\mathcal{P}(q, a)$ is nondegenerate for any nonzero element $0 \neq a \in \mathcal{B}(q)_{-1}$, namely a is nondegenerate by definition. \square

Using (3.3), Lemma 3.2 and [3, Theorem 2.5], we have the following lemma.

Lemma 3.3 *The following conditions on $\Lambda \in \mathcal{B}(q)_0^*$ are equivalent:*

- (1) $L(\Lambda)$ is quasifinite;
- (2) there exists an element $0 \neq a \in \mathcal{B}(q)_{-1}$ such that $\Lambda(\mathcal{B}(q, a)_0) = 0$;
- (3) $M(\Lambda)$ contains a singular vector $a \cdot v_\Lambda \in M(\Lambda)_{-1}$ (cf. Definition 1.1(2)), where $0 \neq a \in \mathcal{B}(q)_{-1}$;
- (4) there exists an element $0 \neq a \in \mathcal{B}(q)_{-1}$ such that $L(\Lambda)$ is an irreducible quotient of the generalized Verma module $M(\mathcal{P}(q, a), \Lambda)$.

Assume that $L(\Lambda)$ is a quasifinite irreducible highest weight module over $\mathcal{B}(q)$. By Lemma 3.3, there exists some monic polynomial $f(t) \in t^q \mathbb{C}[t]$ such that $(x^{-1} f(t)) v_\Lambda = 0$. We shall call such monic polynomial of minimal degree, uniquely determined by the highest weight Λ , the *characteristic polynomial* of $L(\Lambda)$.

Recall that a *quasipolynomial* is a linear combination of functions of the form $p(z)e^{az}$, where $p(z)$ is a polynomial and $a \in \mathbb{C}$. A well-known fact [3, 4, 12–14] stated that a formal power series is a quasipolynomial if and only if it satisfies a nontrivial linear differential equation with constant coefficients.

Proof of Theorem 1.4. Clearly, $f(t)e^{zt} = f(\frac{\partial}{\partial z})e^{zt}$ for $f(t) \in \mathbb{C}[t]$, here and further we use notation $e^{zt} = \sum_{i=0}^{\infty} \frac{z^i}{i!} t^i$ as a generating series of $\mathbb{C}[t]$. For any $f(t) \in t^q \mathbb{C}[t]$, we denote $\tilde{f}(t) := t^{-q} f(t) \in \mathbb{C}[t]$, then $f(t)e^{zt} = \tilde{f}(\frac{\partial}{\partial z})(t^q e^{zt})$. Recalling that the prime stands for the partial derivative $\frac{\partial}{\partial t}$, we have

$$\begin{aligned} \Lambda((f(t)t^q e^{zt})' t^{1-q}) &= \Lambda\left(\left(\tilde{f}\left(\frac{\partial}{\partial z}\right)(t^{2q} e^{zt})\right)' t^{1-q}\right) \\ &= \left(\left(\tilde{f}\left(\frac{\partial}{\partial z}\right)(2qt^{2q-1} e^{zt} + zt^{2q} e^{zt})\right)t^{1-q}\right) \\ &= \tilde{f}\left(\frac{\partial}{\partial z}\right)\Lambda((2q + zt)t^q e^{zt}) = \tilde{f}\left(\frac{d}{dz}\right)\Delta_\Lambda(z, q). \end{aligned} \tag{3.5}$$

If $L(\Lambda)$ is quasifinite, then by (3.2) and Lemma 3.3(2) there exists a polynomial $0 \neq f(t) \in t^q \mathbb{C}[t]$ such that $\Lambda((f(t)g(t))'t^{1-q}) = 0$ for all $g(t) \in t^q \mathbb{C}[t]$. Taking $g(t) = t^q e^{zt}$, by (3.5), we have $\tilde{f}(\frac{d}{dz})\Delta_\Lambda(z, q) = 0$, which implies that $\Delta_\Lambda(z, q)$ is a quasipolynomial.

Conversely, if $\Delta_\Lambda(z, q)$ is a quasipolynomial, then there exists a polynomial $0 \neq h(t) \in \mathbb{C}[t]$ such that $h(\frac{d}{dz})\Delta_\Lambda(z, q) = 0$. Denote $f(t) = t^q h(t) \in t^q \mathbb{C}[t]$, then $\tilde{f}(\frac{d}{dz})\Delta_\Lambda(z, q) = 0$. By (3.5), we have

$$0 = \Lambda((f(t)t^q e^{zt})'t^{1-q}) = \Lambda\left(\left(f(t)\sum_{i=0}^{\infty} \frac{z^i}{i!} t^{q+i}\right)'t^{1-q}\right) = \sum_{i=0}^{\infty} \frac{z^i}{i!} \Lambda((f(t)t^{q+i})'t^{1-q}),$$

which implies $\Lambda((f(t)t^{q+i})'t^{1-q}) = 0$ for $i \in \mathbb{Z}_+$. Hence $\Lambda((f(t)g(t))'t^{1-q}) = 0$ for $g(t) \in t^q \mathbb{C}[t]$ and thus $L(\Lambda)$ is quasifinite by (3.2) and Lemma 3.3(2). \square

4. Intermediate series modules

Suppose $V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu$ is an irreducible uniformly bounded $\mathcal{B}(q)$ -module which is nontrivial as a Vir-module. For any $a \in \mathbb{C}$, we let

$$V[a] = \bigoplus_{\mu \in \mathbb{Z}} V_\mu[a], \quad \text{where } V_\mu[a] = \{v \in V_\mu \mid L_{0,0}v = q(\mu + a)v\}.$$

By (1.2), one can check that $V[a]$ is a $\mathcal{B}(q)$ -submodule, which is a direct summand of V . Thus $V = V[a]$ for some fixed $a \in \mathbb{C}$, namely,

$$V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu, \quad \text{where } V_\mu = V_\mu[a]. \quad (4.1)$$

Note that, regarding as a Vir-module, V is also uniformly bounded. Therefore, by the results of [7, 9, 11], we have the following proposition.

Proposition 4.1 *If V is an irreducible uniformly bounded $\mathcal{B}(q)$ -module as in (4.1), then there exists a non-negative integer N such that $\dim V_\mu[a] = N$ for all $\mu \in \mathbb{Z}$ with $\mu + a \neq 0$.*

The following result is well-known (cf. (1.8)–(1.10)).

Lemma 4.2 *Let $V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu[a]$ be a reducible Vir-module of the intermediate series, then V is isomorphic to one of A_a , B_a or $A'_{0,1} \oplus \mathbb{C}v_0$ as a direct sum of Vir-modules.*

The following lemma seems to be crucial in obtaining Theorem 1.5.

Lemma 4.3 *An irreducible $\mathcal{B}(q)$ -module of the intermediate series V remains to be irreducible when regarded as a Vir -module. In particular, V remains irreducible as a $\mathcal{B}(\frac{q}{k})$ -module for any $k \in \mathbb{Z}_+^*$.*

Proof. The second statement follows from the first since $\text{Vir} \subset \mathcal{B}(\frac{q}{k})$. We prove the first statement in two cases.

Case 1: $q \neq -1$.

If the statement is not true, then there exists a proper irreducible Vir -submodule M . First suppose $M = M_0$ is trivial. Then $L_{\alpha,0}M_0 = 0$, $L_{0,i}M_0 \subset M_0$. Since $\mathcal{B}(q)$ is generated by $\{L_{\alpha,0}, L_{0,i} \mid \alpha, i \in \mathbb{Z}, i \geq 1\}$, we see M is a proper $\mathcal{B}(q)$ -submodule, a contradiction with the irreducibility of V .

Now suppose M is nontrivial, which has to have the form (1.11). Thus $M_\mu := M \cap V_\mu = V_\mu$ if $\mu \neq 0$ and $M_0 = 0$. Then for any $\alpha, \mu \in \mathbb{Z}$ and $i \in \mathbb{Z}_+$,

$$L_{0,i}M_0 = 0, \quad L_{\alpha,0}M_{-\alpha} = 0 \quad \text{and} \quad L_{\alpha,i}M_\mu \subset V_{\alpha+\mu} = M_{\alpha+\mu} \quad \text{if} \quad \mu \neq -\alpha. \quad (4.2)$$

Furthermore, for $\alpha \neq 0$, we have

$$\alpha(q+i)L_{\alpha,i}M_{-\alpha} = [L_{\alpha,0}, L_{0,i}]M_{-\alpha} = L_{\alpha,0}L_{0,i}M_{-\alpha} - L_{0,i}L_{\alpha,0}M_{-\alpha} = 0, \quad (4.3)$$

where the last equality follows from (4.2). Since $q \neq -1$, (4.2) and (4.3) in particular imply $L_{\alpha,i}M \subset M$ for all α, i with $i \leq 1$. Since $\mathcal{B}(q)$ is generated by $\{L_{\alpha,i} \mid \alpha, i \in \mathbb{Z}, 0 \leq i \leq 1\}$, we see M is a nontrivial proper $\mathcal{B}(q)$ -submodule, a contradiction with the irreducibility of V . So, V is an irreducible Vir -module of the intermediate series.

Case 2: $q = -1$.

Suppose V becomes reducible when regarded as a Vir -module, which is isomorphic to A_a , B_a or $A'_{0,1} \oplus \mathbb{C}v_0$ by Lemma 4.2. We should show the following claim, which leads to a contradiction.

Claim 1 *$\mathbb{C}v_0$ is a submodule or a quotient module of V*

Since $\mathcal{B}(-1)$ can be generated by $\{L_{1,1}, L_{0,2}, L_{\alpha,0} \mid \alpha \in \mathbb{Z}\}$, it suffices to determine the actions of $L_{1,1}$ and $L_{0,2}$. Suppose $L_{\alpha,1}v_\mu = e_{\alpha,\mu}v_{\alpha+\mu}$, $L_{\alpha,2}v_\mu = f_{\alpha,\mu}v_{\alpha+\mu}$, and write $e_{1,\mu} = e_\mu$, $f_{0,\mu} = f_\mu$ for short.

Subcase 2.1: $V \cong A_a$ as Vir -modules.

We prove that $e_\mu = 0$ for $\mu \neq 0$, and f_μ is a constant. Then, together with (1.9), it is easy to check that $\mathbb{C}v_0$ is a quotient module of V (thus Claim 1 holds). Applying

$L_{\alpha,1} = [L_{1,1}, L_{\alpha-1,0}]$ to v_μ , we obtain

$$e_{\alpha,\mu} = \begin{cases} (\mu + \alpha)e_\mu - (\mu + \alpha - 1)e_{\mu+\alpha-1} & \text{if } \mu \neq 0, -1, \\ -(\alpha - 1)(a + \alpha - 1)e_{\alpha-1} + \alpha e_0 & \text{if } \mu = 0, \\ -(\alpha - 2)e_{\alpha-2} + (\alpha - 1)(a + \alpha - 1)e_{-1} & \text{if } \mu = -1. \end{cases}$$

Applying $[L_{-1,1}, L_{1,0}] = -L_{0,1}$ to v_0 gives $e_{-1} = 0$. Using this and applying $[L_{\mu,1}, L_{1,0}] = \mu L_{\mu+1,1}$ to v_{-1} with $\mu \neq 1$, we obtain $(\mu - 1)e_{\mu-1} = (\mu - 2)e_{\mu-2}$, which implies that $e_\mu = 0$ for $\mu \neq 0$. Similarly, applying $L_{\alpha,2} = \frac{1}{\alpha}[L_{0,2}, L_{\alpha,0}]$ with $\alpha \neq 0$ to v_μ , we obtain

$$f_{\alpha,\mu} = \begin{cases} \frac{1}{\alpha}(\mu + \alpha)(f_\mu - f_{\alpha+\mu}) & \text{if } \mu \neq 0, \\ (a + \alpha)(f_0 - f_\alpha) & \text{if } \mu = 0. \end{cases}$$

Applying $[L_{\mu,2}, L_{1,0}] = (\mu + 1)L_{\mu+1,2}$ to v_0 with $\mu \neq 0, -1$, and v_{-1} with $\mu \neq 0, \pm 1$ respectively, we obtain

$$\begin{aligned} \text{Ea}_1(\mu) &:= \mu(a + \mu)(f_0 - f_\mu) - (a + 1)(f_1 - f_{\mu+1}) - \mu(a + \mu + 1)(f_0 - f_{\mu+1}) = 0, \\ \text{Ea}_2(\mu) &:= (\mu - 1)(f_{-1} - f_{\mu-1}) - \mu(f_{-1} - f_\mu) = 0. \end{aligned}$$

Solving following linear equations on $f_0, f_{\pm 1}, f_{\pm 2}, f_{\pm 3}$:

$$\begin{cases} \text{Ea}_1(2) = 0, & \text{Ea}_1(-2) = 0, & \text{Ea}_1(-3) = 0, & \text{Ea}_1(1) = 0, \\ \text{Ea}_2(2) = 0, & \text{Ea}_2(-2) = 0, & \text{Ea}_2(-3) = 0, \end{cases}$$

we obtain $f_0 = f_{\pm 1} = f_{\pm 2} = f_{\pm 3}$. Now, rewriting $\text{Ea}_2(\mu) = 0$ as $\mu f_\mu - (\mu - 1)f_{\mu-1} = f_{-1}$, we see that f_μ is a constant. Thus Claim 1 holds.

Subcase 2.2: $V \cong B_a$ as Vir-modules.

We prove that $e_\mu = 0$ for $\mu \in \mathbb{Z}$, and f_μ is a constant. Then, together with (1.10), it is easy to check that $\mathbb{C}v_0$ is a submodule of V (thus Claim 1 holds). Applying $L_{\alpha,1} = [L_{1,1}, L_{\alpha-1,0}]$ to v_μ , we obtain

$$e_{\alpha,\mu} = \begin{cases} (\mu + 1)e_\mu - \mu e_{\alpha+\mu-1} & \text{if } \mu \neq -\alpha, -\alpha + 1, \\ \alpha e_{-1} - (\alpha - 1)(a + \alpha - 1)e_{-\alpha} & \text{if } \mu = -\alpha, \\ (\alpha - 1)(a + \alpha - 1)e_0 - (\alpha - 2)e_{-\alpha+1} & \text{if } \mu = -\alpha + 1. \end{cases}$$

Applying $[L_{\mu,1}, L_{1,0}] = \mu L_{\mu+1,1}$ to $v_{-\mu}$ with $\mu \neq 1$, and $v_{-\mu-1}$ with $\mu \neq 0$ respectively, we

obtain

$$\begin{aligned}\text{Eb}_1(\mu) &:= \mu((\mu+1)e_{-2} - \mu e_{-\mu-1}) - \mu(a+\mu)e_0 + (\mu-1)e_{-\mu} = 0, \\ \text{Eb}_2(\mu) &:= (\mu+1)(\mu e_{-1} - (\mu-1)(a+\mu-1)e_{-\mu}) - (a+1)((\mu+1)e_{-2} - \mu e_{-\mu-1}) \\ &\quad - \mu(\mu(a+\mu)e_0 - (\mu-1)e_{-\mu}) \\ &= 0.\end{aligned}$$

Solving following linear equations on $e_0, e_{-1}, e_{-2}, e_{-3}, e_{-4}$:

$$\begin{cases} \text{Eb}_1(0) = 0, & \text{Eb}_1(2) = 0, & \text{Eb}_1(3) = 0, \\ \text{Eb}_2(1) = 0, & \text{Eb}_2(2) = 0, & \text{Eb}_2(3) = 0, \end{cases}$$

we obtain $e_0 = e_{-1} = e_{-2} = e_{-3} = e_{-4} = 0$. Then $\text{Eb}_1(\mu) = 0$ becomes $(\mu-1)e_{-\mu} = \mu^2 e_{-\mu-1}$, which implies that $e_\mu = 0$ for all $\mu \in \mathbb{Z}$. Similarly, applying $L_{\alpha,2} = \frac{1}{\alpha}[L_{0,2}, L_{\alpha,0}]$ with $\alpha \neq 0$ to v_μ , we obtain

$$f_{\alpha,\mu} = \begin{cases} \frac{1}{\alpha}\mu(f_\mu - f_{\alpha+\mu}) & \text{if } \mu \neq -\alpha, \\ (a+\alpha)(f_0 - f_{-\alpha}) & \text{if } \mu = -\alpha. \end{cases}$$

Applying $[L_{\mu,2}, L_{1,0}] = (\mu+1)L_{\mu+1,2}$ to $v_{-\mu}$ with $\mu \neq 0, \pm 1$, and $v_{-\mu-1}$ with $\mu \neq 0, -1$ respectively, we obtain

$$\begin{aligned}\text{Eb}_3(\mu) &:= (1-\mu)(f_{1-\mu} - f_1) + \mu(f_{-\mu} - f_1) = 0, \\ \text{Eb}_4(\mu) &:= (a+1)(f_{-\mu-1} - f_{-1}) + \mu(a+\mu+1)f_{-\mu-1} - \mu(a+\mu)f_{-\mu} - \mu f_0 = 0.\end{aligned}$$

Solving following linear equations on $f_0, f_{\pm 1}, f_{\pm 2}, f_{\pm 3}$:

$$\begin{cases} \text{Eb}_3(2) = 0, & \text{Eb}_3(-2) = 0, & \text{Eb}_3(3) = 0, \\ \text{Eb}_4(2) = 0, & \text{Eb}_4(-2) = 0, & \text{Eb}_4(1) = 0, & \text{Eb}_4(-3) = 0, \end{cases}$$

we obtain $f_0 = f_{\pm 1} = f_{\pm 2} = f_{\pm 3}$. Now, rewriting $\text{Eb}_3(\mu) = 0$ as $\mu f_{-\mu} - (\mu-1)f_{1-\mu} = f_{-1}$, we see that f_μ is a constant. Thus Claim 1 holds.

Subcase 2.3: $V \cong A'_{0,1} \oplus \mathbb{C}v_0$ as Vir-modules.

For $\mu \neq 0$, we claim that $e_\mu = 0$ and f_μ is a constant. We also claim that $e_0(f_1 - f_0) = 0$. Then, together with (1.11) and the trivial actions of $L_{\alpha,0}$ on v_0 , it is easy to check that $\mathbb{C}v_0$ is a quotient module of V (thus Claim 1 holds). Applying $L_{\alpha,1} = [L_{1,1}, L_{\alpha-1,0}]$ to v_μ , we

obtain

$$e_{\alpha,\mu} = \begin{cases} (\mu + \alpha)e_\mu - (\mu + \alpha - 1)e_{\mu+\alpha-1} & \text{if } \mu \neq 0, -1, \\ \alpha e_0 & \text{if } \mu = 0, \\ -\alpha e_\alpha & \text{if } \mu = -1. \end{cases}$$

Applying $[L_{\mu,1}, L_{1,0}] = \mu L_{\mu+1,1}$ to v_{-1} with $\mu \neq 0, 1$, and v_1 with $\mu \neq -1, -2$ respectively, we obtain

$$\begin{aligned} \text{Ec}_1(\mu) &:= (\mu + 1)e_{\mu+1} - \mu e_\mu = 0, \\ \text{Ec}_2(\mu) &:= (\mu + 1)e_{\mu+1} - \mu e_\mu + e_1 - 2e_2 = 0. \end{aligned}$$

Solving following linear equations on e_{-1}, e_1, e_2, e_3 :

$$\begin{cases} \text{Ec}_1(2) = 0, & \text{Ec}_1(-1) = 0, \\ \text{Ec}_2(2) = 0, & \text{Ec}_2(0) = 0, \end{cases}$$

we obtain $e_{-1} = e_1 = e_2 = e_3 = 0$. Then $\text{Ec}_1(\mu) = 0$ implies $e_\mu = 0$ for $\mu \neq 0$. Similarly, applying $L_{\alpha,2} = \frac{1}{\alpha}[L_{0,2}, L_{\alpha,0}]$ with $\alpha \neq 0$ to v_μ , we obtain

$$f_{\alpha,\mu} = \begin{cases} \frac{1}{\alpha}(\mu + \alpha)(f_\mu - f_{\alpha+\mu}) & \text{if } \mu \neq 0, \\ 0 & \text{if } \mu = 0. \end{cases}$$

Applying $[L_{1,2}, L_{1,0}] = 2L_{2,2}$ to v_{-1} gives $f_{-1} = f_1$; applying $[L_{-3,2}, L_{1,0}] = -2L_{-2,2}$ to v_1 gives $f_{-2} = f_2$. Furthermore, applying $[L_{\mu,2}, L_{1,0}] = (\mu + 1)L_{\mu+1,2}$ to v_{-1} with $\mu \neq 0, \pm 1$, we obtain $\mu f_\mu - (\mu - 1)f_{\mu-1} = f_{-1}$, which implies $f_\mu = f_1$ for $\mu \neq 0$. So the first claim holds. At last, applying $2[L_{1,1}, L_{0,2}] = [[L_{1,1}, L_{-1,2}], L_{1,0}]$ to v_0 gives the second claim. \square

To prove Theorem 1.5(1) and (2) with $q \neq -\frac{1}{2}$, by (1.11), (1.12) and Lemma 4.3, it remains to prove (1.14) (note that (1.13) can be regarded as a special case of (1.14)), which will be done by Lemmas 4.4–4.6. Our philosophy is the following: First we show that $L_{\alpha,1}$ acts as zero for $\alpha \neq 0$. Then for any $(\alpha, i) \neq (0, -2q)$, we can always choose infinite many β 's such that $\beta(2q + i) - \alpha(q + 1) \neq 0$, and so $L_{\alpha,i} = \frac{1}{\beta(2q+i)-\alpha(q+1)}[L_{\alpha-\beta,i-1}, L_{\beta,1}]$ must act as zero, which gives (1.14). To prove Theorem 1.5(3) and (2) with $q = -\frac{1}{2}$, we should make full use of the interesting relations (1.6).

Lemma 4.4 *Suppose $q \neq -\frac{1}{2}, -1$ and we have case (1.12). Then (1.14) holds.*

Proof. Fix $\alpha_0 \neq 0$ and assume $L_{\alpha_0,1}v_\mu = d_\mu v_{\alpha_0+\mu}$ for some $d_\mu \in \mathbb{C}$. We want to prove

$$d_\mu = 0 \text{ for all } \mu \in \mathbb{Z}. \quad (4.4)$$

For convenience, we denote $\mu^+ = \mu + a$ and $\mu^- = \mu - a$. By (1.2), for $\beta, \gamma \in \mathbb{Z}$, we have

$$\begin{aligned} [L_{\gamma,0}, [L_{\beta,0}, L_{\alpha_0,1}]] &= (q(\alpha_0 - \beta) - \beta)(q(\beta + \alpha_0 - \gamma) - \gamma)L_{\gamma+\beta+\alpha_0,1}, \\ [L_{\gamma+\beta,0}, L_{\alpha_0,1}] &= (q(\alpha_0 - \gamma - \beta) - \gamma - \beta)L_{\gamma+\beta+\alpha_0,1}. \end{aligned}$$

Briefly denote the coefficients of right-hand sides by $h_{\beta,\gamma}^{(1)}$ and $h_{\beta,\gamma}^{(2)}$ respectively. Applying the above two equations to v_μ , by (1.12) we obtain

$$qh_{\beta,\gamma}^{(2)}\theta_{\text{lhs}} = h_{\beta,\gamma}^{(1)}\theta_{\text{rhs}}, \quad (4.5)$$

where

$$\begin{aligned} \theta_{\text{lhs}} &= (\beta + \alpha_0 + \mu^+ + b\gamma)((\alpha_0 + \mu^+ + b\beta)d_\mu - (\mu^+ + b\beta)d_{\beta+\mu}) \\ &\quad - (\mu^+ + b\gamma)((\alpha_0 + \mu^+ + \gamma + b\beta)d_{\gamma+\mu} - (\mu^+ + \gamma + b\beta)d_{\gamma+\beta+\mu}), \\ \theta_{\text{rhs}} &= (\alpha_0 + \mu^+ + b(\gamma + \beta))d_\mu - (\mu^+ + b(\gamma + \beta))d_{\gamma+\beta+\mu}. \end{aligned}$$

Now in (4.5), by replacing (γ, β, μ^+) by $(\gamma, \gamma, \beta - \gamma)$, $(\gamma, -\gamma, \beta)$ and $(-\gamma, -\gamma, \beta + \gamma)$ respectively with $\gamma \neq 0$ and $\beta \in a + \mathbb{Z}$, we obtain the following three equations:

$$\begin{cases} (f_{-\gamma,\beta'}^{(1)} - f_{\gamma,\beta'}^{(2)})d_{(\beta-\gamma)^-} + f_{\gamma,\beta}^{(3)}d_{\beta^-} + (f_{-\gamma,\beta}^{(1)} + f_{\gamma,\beta}^{(2)})d_{(\beta+\gamma)^-} = 0, \\ f_{\gamma,\beta}^{(4)}d_{(\beta-\gamma)^-} + f_{\gamma,\beta}^{(5)}d_{\beta^-} + f_{-\gamma,\beta}^{(4)}d_{(\beta+\gamma)^-} = 0, \\ (f_{\gamma,\beta}^{(1)} + f_{-\gamma,\beta}^{(2)})d_{(\beta-\gamma)^-} + f_{-\gamma,\beta}^{(3)}d_{\beta^-} + (f_{\gamma,\beta'}^{(1)} - f_{-\gamma,\beta'}^{(2)})d_{(\beta+\gamma)^-} = 0, \end{cases} \quad (4.6)$$

where $\beta' = \beta + \alpha_0$ and

$$\begin{aligned} f_{x_1,x_2}^{(1)} &= q(q\alpha_0 + 2(1+q)x_1)(bx_1 - x_2)((b-1)x_1 - x_2), \\ f_{x_1,x_2}^{(2)} &= (q\alpha_0 - (1+q)x_1)(q\alpha_0 - x_1)((2b-1)x_1 + x_2), \\ f_{x_1,x_2}^{(3)} &= 2q(q\alpha_0 - 2(1+q)x_1)(\alpha_0 + bx_1 + x_2)((1-b)x_1 - x_2), \\ f_{x_1,x_2}^{(4)} &= q^2\alpha_0(bx_1 - x_2)(\alpha_0 + (b-1)x_1 + x_2), \\ f_{x_1,x_2}^{(5)} &= \alpha_0((1+3q+2q^2(1+b-b^2))x_1^2 + 2q^2(\alpha_0 + x_2)x_2). \end{aligned}$$

for any $x_1, x_2 \in \mathbb{C}$. Regard (4.6) as a system of linear equations on $d_{(\beta-\gamma)^-}$, d_{β^-} , $d_{(\beta+\gamma)^-}$, and let $\Delta_{\beta,\gamma}^{(1)}$ denote the determinant of coefficients, which is a polynomial on β and γ .

Observing that the total degrees on β, γ of $f^{(1)}, f^{(2)}, f^{(3)}$ are ≤ 3 , and those of $f^{(4)}, f^{(5)}$ are ≤ 2 . Hence $\deg \Delta_{\beta, \gamma}^{(1)} \leq 8$. Let $P(i, j)$ denote the coefficient of $\beta^i \gamma^j$ in $\Delta_{\beta, \gamma}^{(1)}$. Direct computation shows

$$\begin{aligned}\Delta_{\beta, \gamma}^{(1)} &= P(0, 8)\gamma^8 + P(1, 6)\beta\gamma^6 + P(0, 6)\gamma^6, \quad \text{where} \\ P(0, 8) &= 8b(1-b)(2b-1)q(1+q)^3(1+2q)\alpha_0, \\ P(1, 6) &= 2(1+q)^2(1+2q)(1+q-2q^2+12bq^2-12b^2q^2)\alpha_0^2.\end{aligned}$$

If $b \neq \frac{1}{2}$, then $P(0, 8) \neq 0$ and thus $\Delta_{\beta, \gamma}^{(1)} \neq 0$, which implies (4.4) holds by (4.6).

If $b = \frac{1}{2}$, then we have

$$P(1, 6)|_{b=\frac{1}{2}} = 2(1+q)^2(1+2q)(1+q+q^2)\alpha_0^2.$$

We use the symbol $\sqrt{-1}$ to stand for the imaginary unit. Then the primitive cube roots of unity can be written as $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, $\omega^2 = -\frac{1}{2} - \frac{\sqrt{-3}}{2}$. Suppose $q \neq \omega, \omega^2$, then $P(1, 6)|_{b=\frac{1}{2}} \neq 0$, which also gives (4.4). Now, suppose $q = \omega$ or ω^2 . Canceling the common term $d_{(\beta+\gamma)-}$ in the first two equations in (4.6) gives an equation (denoted A) on $d_{(\beta-\gamma)-}$ and $d_{\beta-}$. In A, replacing (β, γ) by $(\beta, 1), (\beta-1, 1)$ and $(\beta, 2)$ respectively gives three equations (denoted B) on $d_{(\beta-2)-}, d_{(\beta-1)-}$ and $d_{\beta-}$. In B, canceling the common terms $d_{(\beta-2)-}, d_{(\beta-1)-}$ gives $F_\beta d_{\beta-} = 0$, where F_β is a polynomial on β . Let $H(i)$ denote the coefficient of β^i in F_β . In particular, since $\alpha_0 \in \mathbb{Z}^*$, we have

$$\begin{aligned}H(4)|_{q=\omega} &= -3(1+\sqrt{-3})(24+16\sqrt{-3}+(9-\sqrt{-3}\alpha_0^2))\alpha_0^3 \neq 0, \\ H(4)|_{q=\omega^2} &= -3(1-\sqrt{-3})(24-16\sqrt{-3}+(9+\sqrt{-3}\alpha_0^2))\alpha_0^3 \neq 0,\end{aligned}$$

each of which again implies (4.4) holds. Hence, by our philosophy stated before Lemma 4.4, (1.14) holds. \square

Lemma 4.5 *Suppose $q \neq -\frac{1}{2}, -1$ and we have case (1.11). Then (1.14) holds.*

Proof. Using a similar argument as (4.6), for any $\beta \neq 0, \pm\gamma$ ($\gamma \neq 0$), we have

$$\begin{cases} (g_{-\gamma, \beta'}^{(1)} - g_{\gamma, \beta'}^{(2)})d_{\beta-\gamma} + g_{\gamma, \beta}^{(3)}d_\beta + (g_{-\gamma, \beta}^{(1)} + g_{\gamma, \beta}^{(2)})d_{\beta+\gamma} = 0, \\ g_{\gamma, \beta}^{(4)}d_{\beta-\gamma} + g_{\gamma, \beta}^{(5)}d_\beta + g_{-\gamma, \beta}^{(4)}d_{\beta+\gamma} = 0, \\ (g_{\gamma, \beta}^{(1)} + g_{-\gamma, \beta}^{(2)})d_{\beta-\gamma} + g_{-\gamma, \beta}^{(3)}d_\beta + (g_{\gamma, \beta'}^{(1)} - g_{-\gamma, \beta'}^{(2)})d_{\beta+\gamma} = 0, \end{cases} \quad (4.7)$$

where $g^{(i)} = f^{(i)}|_{b=1}$ for $1 \leq i \leq 5$. Regard (4.7) as a system of linear equations on $d_{\beta-\gamma}$, d_β , $d_{\beta+\gamma}$, and denote $\Delta_{\beta,\gamma}^{(2)}$ the determinant of coefficients, $Q(i, j)$ the coefficient of $\beta^i \gamma^j$ in $\Delta_{\beta,\gamma}^{(2)}$. Then

$$\begin{aligned}\Delta_{\beta,\gamma}^{(2)} &= Q(1, 6)\gamma^6 + Q(0, 6)\gamma^6 \quad \text{where} \\ Q(1, 6) &= 2(1-q)(1+q)^2(1+2q)^2\alpha_0^2, \\ Q(0, 6) &= (1-q)(1+q)^2(1+2q)^2\alpha_0^3.\end{aligned}$$

If $q \neq 1$, then both $Q(1, 6)$ and $Q(0, 6)$ are not equal to zero, and therefore $\Delta_{\beta,\gamma}^{(2)} \neq 0$ for $\beta \neq 0, \pm\gamma$, which implies $d_\mu = 0$ for $\mu \neq 0, \pm\gamma, \pm 2\gamma$ by (4.7). By the arbitrariness of γ , we have $d_\mu = 0$ for all $\mu \in \mathbb{Z}^*$. If $q = 1$, by applying $[L_{0,0}, L_{\alpha_0,1}] = \alpha_0 L_{\alpha_0,1}$ to v_μ , we obtain $\alpha_0 d_\mu = 0$, which also gives $d_\mu = 0$. So (1.14) always holds by our philosophy stated before Lemma 4.4. \square

Lemma 4.6 *Suppose $q = -\frac{1}{2}$. Then (1.14) holds.*

Proof. Recall that $\mathcal{B}(-\frac{1}{4}) \hookrightarrow \mathcal{B}(-\frac{1}{2})$ in the sense that $\mathcal{B}(-\frac{1}{2})$ contains the subalgebra with basis $\{L'_{\alpha,i} = \frac{1}{2}L_{\alpha,2i} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$ isomorphic to $\mathcal{B}(-\frac{1}{4})$. By Lemma 4.3, V remains irreducible when regarded as a Vir-module or a $\mathcal{B}(-\frac{1}{4})$ -module. By Lemmas 4.4 and 4.5, for any $v_\mu \in V_\mu$, $\alpha \in \mathbb{Z}$, we have

$$L_{\alpha,2i}v_\mu = 2L'_{\alpha,i}v_\mu = 0 \quad \text{if } i \geq 1. \quad (4.8)$$

Since $\mathcal{B}(-\frac{1}{2})$ can be generated by $\{L_{0,1}, L_{\alpha,2i} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$, it suffices to determine the action of $L_{0,1}$. Suppose $L_{0,1}v_\mu = e_\mu v_\mu$. We claim that e_μ is a constant (denoted s), which together with (4.8) gives (1.14) with $q = -\frac{1}{2}$.

First suppose we have case (1.12) with $q = -\frac{1}{2}$. For $\alpha \neq 0$, applying $L_{\alpha,1} = \frac{2}{\alpha}[L_{0,1}, L_{\alpha,0}]$ to v_μ , we obtain $L_{\alpha,1}v_\mu = \frac{1}{\alpha}(a + \mu + b\alpha)(e_\mu - e_{\alpha+\mu})v_{\alpha+\mu}$. Furthermore, by (4.8) with $i = 1$, applying $L_{\alpha,2} = \frac{2}{\alpha}[L_{0,1}, L_{\alpha,1}]$ to v_μ gives $\frac{2}{\alpha^2}(a + \mu + b\alpha)(e_\mu - e_{\alpha+\mu})^2 = 0$. In this equation, replacing (α, μ) by $(1, \mu)$, $(-1, \mu + 1)$ and $(2, \mu)$ respectively, we obtain

$$(a + \mu + b)(e_\mu - e_{\mu+1})^2 = 0, \quad (4.9)$$

$$(a + \mu - b + 1)(e_\mu - e_{\mu+1})^2 = 0, \quad (4.10)$$

$$(a + \mu + 2b)(e_\mu - e_{\mu+2})^2 = 0. \quad (4.11)$$

If $b \neq \frac{1}{2}$, by comparing (4.9) and (4.10), then we have $(2b-1)(e_\mu - e_{\mu+1})^2 = 0$, which gives the claim. If $b = \frac{1}{2}$ and $a + \frac{1}{2} \notin \mathbb{Z}$, then the claim still holds by (4.9). If $b = \frac{1}{2}$ and $a + \frac{1}{2} \in \mathbb{Z}$, denoting $\mu_0 = -a - \frac{1}{2}$, then by (4.9) we have, for some $s, s' \in \mathbb{C}$,

$$e_\mu = \begin{cases} s & \text{if } \mu \geq \mu_0 + 1, \\ s' & \text{if } \mu \leq \mu_0. \end{cases} \quad (4.12)$$

On the other hand, taking $\mu = \mu_0$ in (4.11) gives $s = s'$, which again gives the claim.

Now suppose we have case (1.11) with $q = -\frac{1}{2}$. By similar arguments to those in obtaining (4.12), for $\mu \neq 0, -1$, we have $(\mu+1)(e_\mu - e_{\mu+1})^2 = 0$, which gives, for some $s, s' \in \mathbb{C}$,

$$e_\mu = \begin{cases} s & \text{if } \mu \geq 1, \\ s' & \text{if } \mu \leq -2. \end{cases}$$

Applying $L_{-1,2} = 2[L_{-1,1}, L_{0,1}]$ to v_{-2} gives $2(e_{-2} - e_{-1})^2 = 0$, and so $e_{-1} = s'$. Furthermore, applying $L_{2,2} = [L_{0,1}, L_{2,1}]$ to v_{-1} , we obtain $(e_{-1} - e_1)^2 = 0$, and so $s = s'$, which gives the claim. \square

Proof of Theorem 1.5(3). By (1.11), (1.12) and Lemma 4.3, it remains to prove (1.14) with $q = -1$, and (1.15). Similarly to Lemma 4.6, recall the relation $\mathcal{B}(-\frac{1}{2}) \xrightarrow{\neq} \mathcal{B}(-1)$ in the sense that $\mathcal{B}(-1)$ contains the subalgebra with basis $\{L''_{\alpha,i} = \frac{1}{2}L_{\alpha,2i} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$ isomorphic to $\mathcal{B}(-\frac{1}{2})$. By Lemma 4.3, V remains irreducible when regarded as a Vir-module or a $\mathcal{B}(-\frac{1}{2})$ -module. By Lemma 4.6, for any $\alpha \in \mathbb{Z}, i \geq 1$, we have, for some $s \in \mathbb{C}$,

$$L_{\alpha,2i}v_\mu = 2L''_{\alpha,i}v_\mu = \begin{cases} sv_\mu & \text{if } (\alpha, i) = (0, 1), \\ 0 & \text{if } (\alpha, i) \neq (0, 1) \text{ and } i \geq 1. \end{cases} \quad (4.13)$$

Since $\mathcal{B}(-1)$ can be generated by $\{L_{1,1}, L_{\alpha,2i} \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$, it suffices to determine the action of $L_{1,1}$. Suppose $L_{\alpha,1}v_\mu = f_{\alpha,\mu}v_{\alpha+\mu}$, and write $f_{1,\mu} = f_\mu$ for short.

First suppose we have case (1.12) with $q = -1$. We claim that f_μ is a constant (denoted t), which, together with (4.13), gives (1.15). For any $\alpha, \mu \in \mathbb{Z}$, applying $L_{\alpha,1} = [L_{1,1}, L_{\alpha-1,0}]$ to v_μ gives

$$f_{\alpha,\mu} = (a + \mu + 1 + b(\alpha - 1))f_\mu - (a + \mu + b(\alpha - 1))f_{\alpha+\mu-1}.$$

Now applying $[L_{0,1}, L_{\alpha,0}] = 0, [L_{1,1}, L_{\alpha-1,1}] = 0$ to v_μ gives respectively

$$E_1(\alpha, \mu) := (a + \mu + b\alpha)(f_{0,\mu} - f_{0,\alpha+\mu}) = 0;$$

$$E_2(\alpha, \mu) := f_{\alpha-1,\mu}f_{\alpha+\mu-1} - f_{\alpha-1,\mu+1}f_\mu = 0;$$

applying $[L_{\mu,1}, L_{1,0}] = \mu L_{\mu+1,1}$ to v_α gives

$$E_3(\alpha, \mu) := (a + \mu + \alpha + b)f_{\mu,\alpha} - (a + \alpha + b)f_{\mu,\alpha+1} - \mu f_{\mu+1,\alpha} = 0.$$

Solving following equations on $f_{\mu+1}, f_\mu, f_{\mu-1}, f_{\mu-2}$:

$$\begin{cases} E_1(1, \mu) = 0, & E_1(1, \mu - 1) = 0, \\ E_2(0, \mu) = 0, & E_2(3, \mu - 1) = 0, \\ E_3(0, \mu) = 0, & E_3(-1, \mu + 1) = 0, \end{cases} \quad (4.14)$$

we obtain the following possible solutions:

- (i) f_μ is a constant for all $\mu \in \mathbb{Z}$,
- (ii) $b = 0$ and $f_\mu = \begin{cases} 0 & \text{if } \mu \neq -a - 1, \\ t_0 & \text{if } \mu = -a - 1 \text{ (for some } t_0 \neq 0), \end{cases}$
- (iii) $b = 1$ and $f_\mu = \begin{cases} 0 & \text{if } \mu \neq -a, \\ t_1 & \text{if } \mu = -a \text{ (for some } t_1 \neq 0), \end{cases}$

Recall that if $V \cong A_{a,b}$ as a Vir-module, then $b = 0$ or 1 implies $a \notin \mathbb{Z}$. So cases (ii) and (iii) become a special case of (i), and therefore the claim holds.

Suppose we have case (1.11) with $q = -1$. We claim that $f_\mu = 0$ for $\mu \in \mathbb{Z}^*$, which, together with (4.13), gives (1.14) with $q = -1$. Obviously, $f_{-1} = 0$. By similar arguments to those in obtaining (4.14) (or equivalently taking $a = 0, b = 1$ in the last two equations on $f_\mu, f_{\mu-1}$ with $\mu \neq 0, \pm 1$ in (4.14)), the claim holds. \square

Remark 4.7 We can say something more about the irreducible $\mathcal{B}(q)$ -module of the intermediate series for $q \in \frac{1}{2}\mathbb{Z}_-^*$.

- (1) Using the relation $\mathcal{B}(-\frac{1}{2}) \hookrightarrow \mathcal{B}(-\frac{k}{2})$, where $k \in \mathbb{Z}_+^*$, one can deduce that $\mathcal{B}(-\frac{k}{2})$ -module $A'_{0,1}(s)$ is also a $\mathcal{B}(-\frac{1}{2})$ -module $A'_{0,1}(\frac{s}{k})$ for any $s \in \mathbb{C}$;
- (2) Note that $\mathcal{B}(q)$ is perfect if and only if $q \notin \frac{1}{2}\mathbb{Z}_-^*$. So, by Theorem 1.5, an irreducible $\mathcal{B}(q)$ -module of the intermediate series is a trivial extension from an irreducible Vir-module of the intermediate series if and only if $\mathcal{B}(q)$ is perfect.

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